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NONLINEAR HOOKE LAW IN THE THEORY OF ELASTICITY OF INHOMOGENEOUS AND ANISOTROPIC BODIES

Abstract. Directly from the physical connection with the nonlinear Hooke law, the components of the stress tensor of a rigid deformable body and new nonlinear equations of the theory of elasticity with an asymmetric stress tensor are derived, as a special case we obtain equations with the linear Hooke's law. The Lame hypothesis and Lame's equations do not have a physical connection with Hooke's law, this is their falsehood. Lame took as a basis the approximate formula of the incomplete differential and suggested in his hypothesis the proportionality of the stress tensor components to the symmetrical half of the given incomplete differential of displacement, and the antisymmetric half of the differential is discarded, which is the result of the false symmetry of the Lame stress tensor. The new nonlinear equations are approximated by an explicit scheme, with the use of which the elastic state of a flat bar is numerically calculated with the normal and tangential stresses acting on the upper face. The same scheme is applied to the Lame equations. The obtained patterns of displacements distribution clearly demonstrate the difference in the solutions of the comparable systems of elasticity equations, as well as the discrepancy between the solution of the Lame equations for a given state of the deformed body. The falsity of Lame's equations is confirmed theoretically and physically.

Keywords: tensile, tangent, normal, stress, tensor.

1. Tangential stresses according to the generalized Hooke's law

Hooke's law is an assertion according to which the deformation arising

in an elastic body, is proportional to the applied force. It was discovered in 1660 by the English scientist Robert Hooke.

It should be kept in mind that Hooke's law is satisfied only for small deformations. If the proportionality limit is exceeded, the relationship between stresses and strains becomes nonlinear. For many media, Hooke's law does not apply even for small deformations. The derivation of dynamic equations with an asymmetric stress tensor according to Hooke's linear law is given in [1]: $\mathbf{F} = k\mathbf{u}, k>0$, $\mathbf{F}_x = u\mathbf{i}, \mathbf{F}_y = v\mathbf{j}$, $\mathbf{F}_z = w\mathbf{k}$, $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ – vector of displacement, where $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$ – external force that causes displacement.

In inhomogeneous media composed of bodies with various elastic properties or in anisotropic bodies whose properties depend on direction, Hooke's law can be nonlinear

$$\mathbf{F} = k_u u^{m_u}\mathbf{i} + k_v v^{m_v}\mathbf{j} + k_w w^{m_w}\mathbf{k}, \quad k_u > 0, k_v > 0, k_w > 0 \tag{1.1}$$

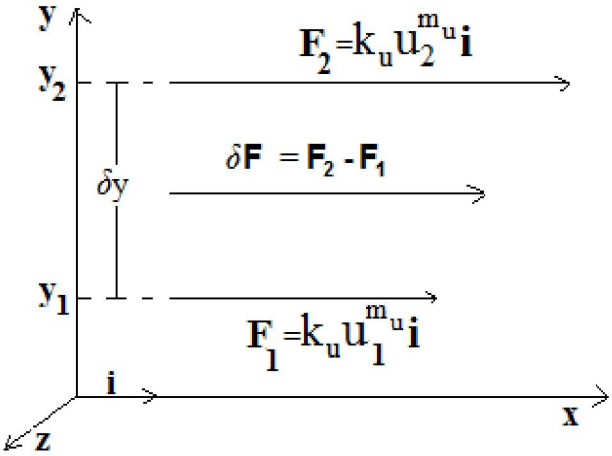
For exponents of 1 and $k_u = k_v = k_w = k$, (1.1) becomes Hooke's linear law for an isotropic medium, hence the exponents must be odd numbers [2], which will be confirmed below by the properties of hyperbolic equations of anguish. Suppose that on the plane y_1 force is associated with the movement

under the generalized law $\mathbf{F}_1 = k_u u_1^{m_u}\mathbf{i}$, similarly force $\mathbf{F}_2 = k_u u_2^{m_u}\mathbf{i}$ on the surface $y_2 = y_1 + \delta y$, $\delta y > 0$.

The increments of forces and displacements between layers are equal:

$$\delta\mathbf{F} = \mathbf{F}_2 - \mathbf{F}_1 = k_u u_2^{m_u}\mathbf{i} - k_u u_1^{m_u}\mathbf{i} = k_u \delta u^{m_u}\mathbf{i},\ \delta u^{m_u} = u_2^{m_u} - u_1^{m_u} > 0.$$

Let $|\mathbf{F}_2| > |\mathbf{F}_1|$, in this case the force increment is directed along the x axis: $\delta\mathbf{F} \uparrow\uparrow \mathbf{i}$.



Linear density is introduced $\mathbf{f} = \delta\mathbf{F}/\delta y,\ \delta\mathbf{F} = \delta y\mathbf{f}$. By definition, the average tangent stress vector $\mathbf{p}_{yxav} = \dfrac{\delta\mathbf{F}}{\delta x \delta z}$ parallel and equally directed with a force that causes this voltage $\mathbf{p}_{yxav} \uparrow\uparrow \delta\mathbf{F}$, $\mathbf{p}_{yxav} \uparrow\uparrow \mathbf{f}$.

By the introduction of the proportionality coefficient, the following bonds are formed:

$$\mathbf{f} = k'\mathbf{p}_{yxav}, k'>0,\ \delta y\mathbf{f} = k'\mathbf{p}_{yxav}\delta y,\ \mathbf{p}_{yxav} \uparrow\uparrow \mathbf{i},$$

$$k'\mathbf{p}_{yxav}\delta y = k_u \delta u^{m_u}\mathbf{i}$$

This expression is multiplied scalar by the unit vector $\mathbf{i}$:

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = (k_u \delta u^{m_u}\mathbf{i}, \mathbf{i})$$

As result

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = k'|\mathbf{p}_{yxav}|\delta y|\mathbf{i}|\cos 0^o = k'p_{yxav}\delta y,\ (k_u \delta u^{m_u}\mathbf{i}, \mathbf{i}) = k_u \delta u^{m_u}$$

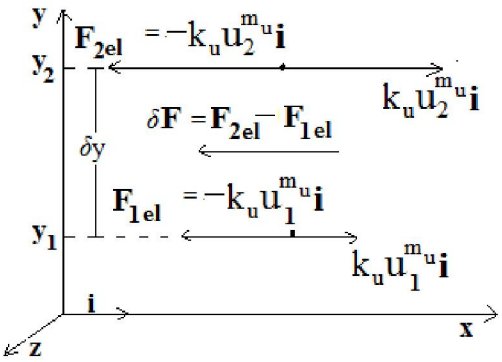
Equalities $k'p_{yxav}\delta y = k_u \delta u^{m_u}$, $p_{yxav} = \dfrac{k_u}{k'}\dfrac{\delta u^{m_u}}{\delta y}$ in the limit give a tangential stress

$$p_{yx} = \lim_{\delta y \to 0}\frac{k_u}{k'}\frac{\delta u^{m_u}}{\delta y} = \mu_u \frac{\partial u^{m_u}}{\partial y},\ \mu_u = \frac{k_u}{k'} > 0$$

Tangential stresses in other directions are obtained analogously:

$$p_{xy} = \mu_v \frac{\partial v^{m_v}}{\partial x}, p_{zx} = \mu_u \frac{\partial u^{m_u}}{\partial z}, p_{xz} = \mu_w \frac{\partial w^{m_w}}{\partial x}, p_{yz} = \mu_w \frac{\partial w^{m_w}}{\partial y}, p_{zy} = \mu_v \frac{\partial v^{m_v}}{\partial z}$$

The asymmetric shearing stress formulas are derived for causing stretching of the body of an external force $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$.



The elastic force in the deformed body according to Newton's third law is equal to the external force with a minus sign $\mathbf{F}_{el} = -\mathbf{F}$. Consequently, the linear Hooke law for elastic forces will have the form $\mathbf{F}_{el} = -k\mathbf{u}, \mathbf{F}_{el} = -ku\mathbf{i} - kv\mathbf{j} - kw\mathbf{k}$. A similar representation for Hooke's nonlinear law

$$\mathbf{F}_{el} = -k_u u^{m_u}\mathbf{i} - k_v v^{m_v}\mathbf{j} - k_w w^{m_w}\mathbf{k}$$

Suppose that on the plane y1 the force is connected with the displacement along the nonlinear law $\mathbf{F}_{1el} = k_u u_1^{m_u}\mathbf{i}$, similarly acts force $\mathbf{F}_{2el} = k_u u_2^{m_u}\mathbf{i}$ on surface $y_2 = y_1 + \delta y,\ \delta y > 0$.

The increments of forces and displacements between layers are equal:

$$\delta\mathbf{F} = \mathbf{F}_{2el} - \mathbf{F}_{1el} = -k_u u_2^{m_u}\mathbf{i} + k_u u_1^{m_u}\mathbf{i} = -k_u \delta u^{m_u}\mathbf{i},$$
$$\delta u^{m_u} = u_2^{m_u} - u_1^{m_u} > 0.$$

Let $|\mathbf{F}_2| > |\mathbf{F}_1|$, in this case, the force increment is directed against the x axis: $\delta\mathbf{F} \uparrow\downarrow \mathbf{i}$. Linear density is introduced $\mathbf{f} = \delta\mathbf{F}/\delta y,\ \delta\mathbf{F} = \delta y\mathbf{f}$.

By definition, the average tangent stress vector $\mathbf{p}_{yxav} = \dfrac{\delta\mathbf{F}}{\delta\sigma}, \delta\sigma = \delta x\delta z$ parallel and equally directed with a force that causes this voltage $\mathbf{p}_{yxav} \uparrow\uparrow \delta\mathbf{F},\ \mathbf{p}_{yxav} \uparrow\uparrow \mathbf{f}$.

The input of the proportionality coefficient formed a bond:

$$\mathbf{f} = k'\mathbf{p}_{yxav}, k'>0,\ \delta y\mathbf{f} = k'\mathbf{p}_{yxav}\delta y,\ \mathbf{p}_{yxav} \uparrow\downarrow \mathbf{i},\ k'\mathbf{p}_{yxav}\delta y = -k_u\delta u^{m_u}\mathbf{i}$$

This expression is multiplied scalar by the unit vector $\mathbf{i}$:

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = -(k_u\delta u^{m_u}\mathbf{i}, \mathbf{i})$$

As a result

$$(k'\mathbf{p}_{yxav}\delta y, \mathbf{i}) = k'|\mathbf{p}_{yxav}|\delta y|\mathbf{i}|\cos 180^o = -k'p_{yxav}\delta y,\ -(k_u\delta u^{m_u}\mathbf{i}, \mathbf{i}) = -k_u\delta u^{m_u}$$

Equalities

$$k'p_{yxav}\delta y = k_u\delta u^{m_u},\ p_{yxav} = \frac{k_u}{k'}\frac{\delta u^{m_u}}{\delta y}$$

in the limit give a tangential stress

$$p_{yx} = \lim_{\delta y \to 0}\frac{k_u}{k'}\frac{\delta u^{m_u}}{\delta y} = \mu_u\frac{\partial u^{m_u}}{\partial y},\ \mu_u = \frac{k_u}{k'} > 0$$

Similarly, tangential stresses are obtained in other directions:

$$p_{xy}=\mu_v\frac{\partial v^{m_v}}{\partial x},p_{zx}=\mu_u\frac{\partial u^{m_u}}{\partial z},p_{xz}=\mu_w\frac{\partial w^{m_w}}{\partial x},p_{yz}=\mu_w\frac{\partial w^{m_w}}{\partial y},p_{zy}=\mu_v\frac{\partial v^{m_v}}{\partial z}$$

In this way, the stress formulas for external forces coincide with formulas of the action of elastic forces, therefore further conclusions are made only for external forces.

2. Relation of Normal Stresses to Hooke's Law

A similar argument establishes the formula of the component $\mathbf{p}^{o}_{xx}$ normal voltage $\mathbf{p}_{xx}=\lambda \mathrm{div}\mathbf{u}\ \mathbf{i}+\mathbf{p}^{o}_{xx}$.

Let the external forces be equal: $\mathbf{F}_1=k_u u_1^{m_u}\mathbf{i}$ on surface x_1 and $\mathbf{F}_2=k_u u_2^{m_u}\mathbf{i}$ on surface $x_2=x_1+\delta x,\delta x>0.$

The increments of forces and displacements between layers are equal:

$$\delta\mathbf{F}=\mathbf{F}_2-\mathbf{F}_1=k_u u_2^{m_u}\mathbf{i}-k_u u_1^{m_u}\mathbf{i}=k_u\delta u^{m_u}\mathbf{i},\ \delta u^{m_u}=u_2^{m_u}-u_1^{m_u}>0.$$

By definition, the average stress vector $\mathbf{p}^{o}_{xx\,av}=\frac{\delta\mathbf{F}}{\delta\sigma},\delta\sigma=\delta y\delta z$ parallel and equally directed with a force that causes this voltage $\delta\mathbf{F}\uparrow\uparrow\mathbf{i}$, $|\mathbf{F}_2|>|\mathbf{F}_1|$.

Through the linear density $\mathbf{f}=\frac{\delta\mathbf{F}}{\delta x},\delta\mathbf{F}=\delta x\mathbf{f},\ \mathbf{f}=k''\mathbf{p}^{0}_{xx\,av}$ equalities $\delta\mathbf{F}=k''\delta x\mathbf{p}^{0}_{xx\,av}$, $k''\delta x\mathbf{p}^{0}_{xx\,av}=k_u\delta u^{m_u}\mathbf{i}.$

This expression is multiplied scalar by the unit vector $\mathbf{i}$:

$$(k''\delta x\mathbf{p}^{0}_{xx\,av},\mathbf{i})=(k_u\delta u^{m_u}\mathbf{i},\mathbf{i})$$

Vectors are parallel in structure $\mathbf{p}^{0}_{xx\,cp}\uparrow\uparrow\mathbf{i}.$

Therefore, they occur in scalar products

$$(k''\delta x\mathbf{p}^{0}_{xxav},\mathbf{i})=k''\delta x\left|\mathbf{p}^{0}_{xxav}\right|\cdot|\mathbf{i}|\cdot\cos 0^{o}=k''\delta x p^{0}_{xxav},\ (k_u\delta u^{m_u}\mathbf{i},\mathbf{i})=k_u\delta u^{m_u}.$$

The result is $k''\delta x p^{0}_{xxav}=k_u\delta u^{m_u}$, whence $p^{0}_{xxav}=\frac{k_u}{k''}\frac{\delta u^{m_u}}{\delta x}$.

In the limit, the formula for the component of the normal stress

$$p^{0}_{xx}=\lim_{\delta x\to 0}\frac{k_u}{k''}\frac{\delta u^{m_u}}{\delta x}=\mu_u\frac{\partial u^{m_u}}{\partial x},\ \mu_u=\frac{k_u}{k''}.$$

The same reasonings also show the components of normal stresses in other direc tions:

$$p^{o}_{ii}=\mu_i\frac{\partial u_i^{m_i}}{\partial x_i},\ \mu_i=\frac{k_i}{k''},i=1,2,3;\ u_1\equiv u,u_2\equiv v,u_3\equiv w,x_1\equiv x,x_2\equiv y,x_3\equiv z.$$