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D.S. Dzhumabaev[1], A.S. Zharmagambetov [2]

[1,2]International Information Technologies University, Almaty, Kazakhstan,
Department of Differential Equations, Institute of Mathematics and Mathematical Modeling of MES RK, 125, Pushkin str., Almaty, Kazakhstan[1]
dzhumabaev@list.ru[1] armanform@gmail.com[2]

NUMERICAL METHOD FOR SOLVING A LINEAR BOUNDARY VALUE PROBLEM FOR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

Annotation. Numerical method to solve the linear boundary value problem for the Fredholm integro-differential equations with degenerate kernel is proposed. Dividing interval into N parts and introducing additional parameters as the values of solution at the left-end points of subintervals origin problem is reduced to the multipoint boundary value problem for the system of integro-differential equations with parameters. At the fixed values of parameters the special Cauchy problems for the system of integro-differential equations are solved. Introducing the additional parameters allows as the solvability of the boundary value problem to reduce to the solvability of system of linear algebraic equation with respect to introduced parameters. The Cauchy problems for the ordinary differential equations and evaluating the definite integrals on subintervals are the main auxiliary problems of method proposed. The Cauchy problems are solved by the Bulirsch-Stoer method and definite integrals are determined by the Sympson method.

Keywords. Fredholm integro-differential equations, linear boundary value problem, parametrization, Bulirsch-Stoer method.

Various problems of physics, engineering, biology, etc. lead to the study of integro-differential equations and to the formulation of related specific tasks. In connection with this, the theory of such equations has attracted the attention of mathematicians. Qualitative properties of problems for the Fredholm integro-differential equations and methods for solving these problems are considered in the works of many others [1-11].

In the present paper, we consider the linear boundary value problem for Fredholm integro-differential equation:

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \int_0^T \varphi_k(t)\psi_k(s)x(s)ds + f(t), t \in (0, T), x \in R^n \tag{1}$$

$$Bx(0) + Cx(T) = d, d \in R^n \tag{2}$$

where the matrices A(t), $\varphi_k(t)$, $\psi_k(s)$, $0<k<m$ and vector f(t) are continuous on [0, T]. A solution to problem (1) and (2) is a vector function x(t), continuous on [0, T] and continuously differentiable on (0, T), satisfying the integro-differential equation (1) and boundary condition (2).

Given the points: $t_0 = 0 < t_1 < \ldots < t_N = T$, and let Δ_N denote the partition of interval [0, T) into N subintervals [0, T) = $\bigcup_{r=1}^{N} [t_{r-1}, t_r]$. The case, when the interval [0, T] is not divided into parts, we denote by Δ_1.

Let $x_r(t)$ be the restriction of function x(t) to the r -th interval $[t_{r-1}, t_r)$, i.e. $x_r(t) = x(t)$, for $t \in [t_{r-1}, t_r)$, r = 1,N.

Introducing the additional parameters $\lambda_r = x_r(t_{r-1})$ and performing a replacement of the function $u_r(t) = x_r(t) - \lambda_r$ on each r -th interval, we obtain the following boundary value problem with parameters:

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{j=1}^{N}\sum_{k=1}^{m}\int_{t_{j-1}}^{t_j} \varphi_k(t)\psi_k(s)(u_r(s) + \lambda_r)ds + f(t) \tag{3}$$

$$t \in (t_{r-1}, t_r),\ r = \overline{1, N},$$

$$u_r(t_{r-1}) = 0, r = 1...N \tag{4}$$

$$B\lambda_1 + C\lambda_N + C\lim_{t\to T-0} u_N(t) = d \tag{5}$$

$$\lambda_p + \lim_{t\to t_p-0} u_p(t) - \lambda_{p+1} = 0, p = 1...N-1 \tag{6}$$

where (6) are conditions for matching the solution at the interior points of the partition Δ_N. Note, that conditions (6) and integro-differential equations (3) also ensure the continuity of solution's derivatives at these points.

Using the fundamental matrix $X_r(t)$ of differential equation dx/dt = A(t)x on $[t_{r-1}, t_r]$, we reduce the special Cauchy problem for the system of integro-differential equations with parameters (2.3), (2.4) to the equivalent system of integral equations.

$$u_r(t) = X_r\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)d\tau\lambda_r +$$

$$+X_r\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)\sum_{j=1}^{N}\sum_{k=1}^{m}\int_{t_{j-1}}^{t_j} \varphi_k(t)\psi_k(s)(w(s)_r + \lambda_r)dsd\tau + \tag{7}$$

$$+X_r\int_{t_{r-1}}^{t} X_r^{-1}(\tau)A(\tau)f(\tau)d\tau, t \in [t_{r-1}, t_r), r = 1...N$$

Introduce the notation

$$\mu_k = \sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \psi_k(s)u_j(s)ds$$

Multiplying both sides of (7) by $\psi_p(t)$, integrating on the interval $[t_{r-1}, t_r]$ and summing up over r, we have the system of linear algebraic equations with respect to $\mu = (\mu_1, \ldots, \mu_m) \in R^{nm}$:

$$\mu_p = \sum_{k=1}^{m} G_{p,k}(\Delta_N)\mu_k + \sum_{r=1}^{N} V_{p,r}(\Delta_N)\lambda_r + g_p(f, \Delta_N), p = 1...m \tag{8}$$

with the (n × n) matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau) \phi_k(s) ds d\tau \tag{9}$$

$$V_{p,r}(\Delta_N) = \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau) A(s) ds d\tau +$$

$$+ \sum_{j=1}^{N} \sum_{k=1}^{m} \int_{t_{j-1}}^{t_j} \psi_p(\tau) X_j(\tau) \int_{t_{j-1}}^{\tau} X_j^{-1}(\tau_1) \phi_k(\tau_1) d\tau_1 d\tau \int_{t_{r-1}}^{t_r} \psi_k(s) ds \tag{10}$$

and vectors of dimension n

$$g_p(f, \Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) X_r(\tau) \int_{t_{r-1}}^{\tau} X_r^{-1}(\tau) f(s) ds d\tau \tag{11}$$

Using these matrices we can rewrite the system (8) in the form

$$[I - G(\Delta_N)]\mu = V(\Delta_N)\lambda + g(f, \Delta_N), \tag{12}$$

where I is the identity matrix of dimension nm.

Essential requirement to the partition is its regularity. Partition Δ_N is called regular if the matrix $I - G(\Delta_N)$ is invertible. It is established that the invertability of system's matrix is equivalent to the well-posedness of considered boundary value problem [2]. Assume the matrix $I - G(\Delta_N)$ is invertible and $[I - G(\Delta_N)]^{-1} = \left(M_{k,p}(\Delta_N)\right)$. Then according to (12) the elements of the vector $\mu \in R^{nm}$ are determined by the equalities

$$\mu_k = \sum_{j=1}^{N} \left(\sum_{p=1}^{m} M_{k,p}(\Delta_N) V_{p,r}(\Delta_N)\right) \lambda_j + \sum_{p=1}^{m} M_{k,p}(\Delta_N) g_p(f, \Delta_N), \tag{13}$$

Substituting the right-hand side of (13) instead of μ_k, we get the representation of functions $u_r(t)$ via λ_j

$$u_r(t) = \sum_{j=1}^{N} \left\{ \sum_{k=1}^{m} X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau) \phi_k(\tau) \right.$$

$$\times \left[\sum_{p=1}^{m} M_{k,p}(\Delta_N) V_{p,j}(\Delta_N) + \int_{t_{j-1}}^{t_j} \psi_k(s) ds \right] \lambda_j +$$

$$+ X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau) A(\tau) d\tau \lambda_r + X_r(t) \int_{t_{r-1}}^{t} X_r^{-1} \times$$

$$\times \left[\sum_{k=1}^{m} \phi_k(\tau) \sum_{p=1}^{m} M_{k,p}(\Delta_N) g_p(f, \Delta_N) + f(\tau) \right] d\tau, \tag{14}$$

Introduce notation

$$D_{r,j}(\Delta_N) = \sum_{k=1}^{2m} X_r(t) \int_{t_{r-1}}^{t} X_r^{-1}(\tau)\phi_k(\tau) \times$$

$$\times \left[\sum_{p=1}^{2m} M_{k,p}(\Delta_N) V_{p,j}(\Delta_N) + \int_{t_{j-1}}^{t_j} \psi_k(s) ds \right] \tag{15}$$

$$D_{r,r}(\Delta) = \sum_{k=1}^{2m} X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau)\phi_k(\tau) d\tau \times$$

$$\times \left[\sum_{p=1}^{2m} M_{k,p}(\Delta_N) V_{p,j}(\Delta_N) + \int_{t_{j-1}}^{t_j} \psi_k(s) ds \right] + \tag{16}$$

$$+ X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) A(\tau) d\tau$$

$$F_r(\Delta_N) = \sum_{k=1}^{2m} X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) A(\tau) d\tau \sum_{p=1}^{2m} M_{k,p}(\Delta_N) g_p(f, \Delta_N) + \tag{17}$$

$$+ \sum_{k=1}^{2m} X_r(t_r) \int_{t_{r-1}}^{t_r} X_r^{-1}(\tau) f(\tau) d\tau, r = 1 \ldots N,$$

Then from (14) we have

$$\lim_{t \to t_r - 0} u_r(t) = \sum_{j=1}^{N} D_{r,j}(\Delta_N)\lambda_j + F_r(\Delta_N). \tag{18}$$

Substituting the right-hand side of (18) into the boundary condition (5) and conditions of matching solution (6), we obtain the following system of linear algebraic equations with respect to parameters λ_r, r = (1,N) :

$$[B + CD_{N,1}(\Delta_N)]\lambda_1 + \sum_{j=2}^{N-1} CD_{N,j}(\Delta_N)\lambda_j +$$

$$+ C[I + D_{N,N}(\Delta_N)]\lambda_N = d - CF_N(\Delta_N), \tag{19}$$

$$[I + D_{p,p}(\Delta_N)]\lambda_p - [I + D_{p,p+1}(\Delta_N)]\lambda_{p+1} +$$

$$+ \sum_{j=1}^{N} D_{p,j}(\Delta_N)\lambda_j = -F_p(\Delta_N), p = 1 \ldots N-1 \tag{20}$$

By denoting the matrix corresponding to the left-hand side of the system of equations (19), (20) by Q, the system can be written as the following:

$$Q_*(\Delta_N)\lambda = -F_*(\Delta_N), \lambda \in R^{nN}, \tag{21}$$

where