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ANALYTICAL SOLUTION OF HEAT EQUATION BY HEAT POLYNOMIALS

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Abstract. Solution of heat equation with second type boundary conditions represented in explicit analytical form. The developed method is based on use of Integral Error Functions and its properties which enables to solve heat and mass transfer problems with domains that include moving boundaries. Elaborated method can be effectively used in the fields of engineering, which require consideration of phenomena with phase transformations, such as heat and mass transfer problems, low temperature plasma, filtration mathematical models of which are based on Stefan type problems. The main idea of this method is to find coefficients of Heat Polynomial which a priori satisfy the heat equation.

АНАЛИТИЧЕСКОЕ РЕШЕНИЕ УРАВНЕНИЯ ТЕПЛОПРОВОДНОСТИ С ПОМОЩЬЮ ТЕПЛОВЫХ ПОЛИНОМОВ

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Ключевые слова. Аналитическое решение, уравнения, теплопроводность, границы.

Аннотация. Найдено аналитическое решение уравнения теплопроводности в областях с подвижными границами, вырождающимимся в начальный момент времени с помощью тепловых полиномов.

1.Introduction:

Despite the quite extensive literature on various types of moving boundary value problems both in theoretical and numerical aspects; see, e.g., [1-5] and a long bibliography on these problems [6] we are still not able to apply offered methods for solving Stefan type Problems particularly necessary for mathematical modelling of arc phenomena in electrical contacts, solution of which originally based on the reduction to the systems of integral equations [7], and accepted non degenerate at the initial time [3],[5] or at least give qualitative [8] solution which is inapplicable in above mentioned engineering problems.

The aim of this paper is to find both qualitative and quantitative solution of heat equation in domain with moving boundary that degenerate at the initial time and which can be helpful for solution of heat and mass transfer problems that include phase transformations.

Tracking answers of these questions will be organized as following. In the first section some necessary properties of Integral Error Function that are used for solving heat equation with moving boundaries are represented. In the second section by the use of multinomial coefficients of Newton's polynomials solution of heat equation with second type boundary conditions is developed. The third section is devoted for conclusion and discussion.

1.1 Integral Error Functions

Heat equations are solved by the help of so called IEF method (Integral Error Functions or Hartree functions method) and properties of Integral Error Functions which were introduced by Hartree in 1935 and reasonably sometimes called Hartree functions.

The integral error functions were determined by recurrent formulas

$$i^n erfcx = \int_x^\infty i^{n-1} erfcv dv \quad , \quad n=1,2,\ldots \qquad i^0 erfcx \equiv erfcx = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-v^2) dv \tag{1}$$

where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv \tag{2}$$

One can obtain from

$$i^n erfcx = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_x^\infty (v-x)^n \exp(-v^2) dv \tag{3}$$

Expressions (1) satisfy the differential equation

$$\frac{d^2}{dx^2} i^n erfcx + 2x \frac{d}{dx} i^n erfcx - 2n i^n erfcx = 0 \tag{4}$$

and recurrent formulas

$$2n i^n erfcx = i^{n-2} erfcx - 2x i^{n-1} erfcx \tag{5}$$

Integral Error Functions are very useful for investigation of heat transfer, diffusion and other phenomena which can be described by the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6}$$

region $D(t > 0, 0 < x < \alpha(t))$ with free boundary $x = \alpha(t)$, since the functions

$$u_n(\pm x, t) = t^{\frac{n}{2}} i^n erfc \frac{\pm x}{2a\sqrt{t}}$$

suffice the equation (6) as well as their linear combination or even series

$$u(x,t) = \sum_{n=0}^{\infty} [A_n u_n(x,t) + B_n u_n(-x,t)]$$

For any constants A_n, B_n.We can choose these constants to satisfy the boundary conditions at $x=0$ and $x=a(t)$, if given boundary functions can be expanded into Maclaurin series with powers t or $\sqrt{t}$.

1.2 Properties of Integral Error Functions

It is possible to derive properties of Integral Error Functions.

If n is an integer, then

$$i^n erfc(-x) + (-1)^n i^n erfcx = \frac{1}{2^{n-1} n! i^n} H_n(ix) = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n} e^{x^2}$$

with $i = \sqrt{-1}$ and Hermite polynomials $H_n(x)$ in the right side. Indeed, using formula (1) one can write

$$i^n erfc(-x) + (-1)^n i^n erfcx = \frac{2}{\sqrt{\pi}} \frac{1}{n!} \int_{-x}^\infty (v+x)^n \exp(-v^2) dv +$$

$$\frac{(-1)^n 2}{n!\sqrt{\pi}} \int_x^\infty (v-x)^n \exp(-v^2) dv = \frac{2}{n!\sqrt{\pi}} \int_{-\infty}^\infty (v+x)^n \exp(-v^2) dv = \frac{1}{2^{n-1} n! i^n} H_n(ix) \tag{7}$$

Using formula for Hermite polynomials one can derive

$$i^n erfc(-x) + (-1)^n i^n erfcx = \sum_{m=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2m}}{2^{2m-1} m!(n-2m)!} \tag{8}$$

If $n = 2k$, then

$$i^{2k} erfc\ x + i^{2k} erfc\ (-x) = \sum_{m=0}^{k} \frac{x^{2(k-m)}}{2^{2m-1} m!(2k-2m)!}$$

In particular

$$erfc\ x + erfc\ (-x) = 2,$$

$$i^2 erfc\ x + i^2 erfc\ (-x) = \frac{1}{2} + x^2,$$

$$i^4 erfc\ x + i^4 erfc\ (-x) = \frac{1}{8} + \frac{1}{4}x^2 + \frac{1}{12}x^4.$$

If $n = 2k+1$, then

$$i^{2k+1} erfc(-x) - i^{2k+1} erfc\ x = \sum_{m=0}^{k} \frac{x^{2(k-m)+1}}{2^{2m-1} m!(2k-2m+1)!} \tag{9}$$

In particular

$$ierfc(-x) - ierfc\ x = 2x,$$

$$i^3 erfc(-x) - i^3 erfc\ x = \frac{1}{2}x + \frac{1}{3}x^3,$$

$$i^5 erfc\ (-x) - i^5 erfc\ x = \frac{1}{2^3 \cdot 2!}x + \frac{1}{2 \cdot 2! \cdot 3!}x^3 + \frac{2}{5!}x^5.$$

The proof of the formula

$$i^n erfc(-x) - (-1)^n i^n erfcx = \frac{1}{2^{n-1} n!} e^{-x^2} \frac{d^n}{dx^n}(e^{x^2} erfx) \tag{10}$$

where

$$erfx = 1 - erfcx = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-v^2) dv$$

can be obtained by mathematical induction method using recurrent formula (5).

(11)

Using L'Hopital rule and representation (1), it is not difficult to show that

$$\lim_{x \to \infty} \frac{i^n erfc(-x)}{x^n} = \frac{2}{n!} \tag{12}$$

Using property 2 one can derive following formula

$$u(x,t) = \sum_{n=0}^{\infty} \left[A_{2n} \sum_{m=0}^{n} x^{2n-2m} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} x^{2n-2m+1} t^m \beta_{2n+1,m} \right]$$

Where u(x,t) is Heat polynomial which exactly satisfy Heat Equation where $\beta_{n,m} = \frac{1}{2^{n+m-1} m!\,(n-2m)!}$

2. Problem Statement

2.1 It is required to find the solution of Heat Equation with moving (known) boundary that degenerate at the initial time

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \alpha(t), \qquad t > 0 \tag{13}$$

where $\alpha(t) = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots + \alpha_k t^k + \cdots$ (14)

$I.C: \quad u(x,0) = 0$ (15)

$B.C: \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = \psi(t)$ (16)

$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} = \varphi(t)$ (17)

From property (4) section 1.2 we consider solution in the form of Heat Polynomials

$$u(x,t) = \sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n} x^{2n-2m}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n} x^{2n-2m+1}t^m\beta_{2n+1,m}\right] \quad (18)$$

2.2 Method of solution

$$\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)x^{2n-2m-1}t^m\beta_{2n,m} + A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)x^{2n-2m}t^m\beta_{2n+1,m}\right] \equiv$$

$$\begin{aligned}
&\equiv A_2 2x\beta_{2,0} + \\
&A_4\left(4x^3\beta_{4,0} + 2xt\beta_{2,1}\right) + \\
&A_6\left(6x^5\beta_{6,0} + 4x^3t\beta_{6,1} + 2xt^2\beta_{6,2}\right) + \\
&A_8\left(8x^7\beta_{8,0} + 6x^5t\beta_{8,1} + 4x^3t^2\beta_{8,2} + 2xt^3\beta_{8,3}\right) + \\
&A_{10}\left(10x^9\beta_{10,0} + 8x^7t\beta_{10,1} + 6x^5t^2\beta_{10,2} + 4x^3t^3\beta_{10,3} + 2xt^4\beta_{10,4}\right) + \\
&A_{12}\left(12x^{11}\beta_{12,0} + 10x^9t\beta_{12,1} + 8x^7t^2\beta_{12,2} + 6x^5t^3\beta_{12,3} + 4x^3t^4\beta_{12,4} + 2xt^5\beta_{12,5}\right) + \\
&A_{14}\left(14x^{13}\beta_{14,0} + 12x^{11}t\beta_{14,1} + 10x^9t^2\beta_{14,2} + 8x^7t^3\beta_{14,3} + 6x^5t^4\beta_{14,4} + 4x^3t^5\beta_{14,5}\right. \\
&\qquad \left. + 2xt^6\beta_{14,6}\right) + \cdots \\
&\ldots + A_{2k}\left(2kx^{2k-1}\beta_{2k,0} + (2k-1)x^{2k-3}t\beta_{2k,1} + \cdots + 2xt^{k-1}\beta_{2k,k-1}\right) + \cdots + \\
&A_1\beta_{1,0} + \\
&A_3\left(3x^2\beta_{3,0} + t\beta_{3,1}\right) + \\
&A_5\left(5x^4\beta_{5,0} + 3x^2t\beta_{5,1} + t^2\beta_{5,2}\right) + \\
&A_7\left(7x^6\beta_{7,0} + 5x^4t\beta_{7,1} + 3x^2t^2\beta_{7,2} + t^3\beta_{7,3}\right) + \\
&A_9\left(9x^8\beta_{9,0} + 7x^6t\beta_{9,1} + 5x^4t^2\beta_{9,2} + 3x^2t^3\beta_{9,3} + t^4\beta_{9,4}\right) + \\
&A_{11}\left(11x^{10}\beta_{11,0} + 9x^8t\beta_{11,1} + 7x^6t^2\beta_{11,2} + 5x^4t^3\beta_{11,3} + 3x^2t^4\beta_{11,4} + t^5\beta_{11,5}\right) + \\
&A_{13}\left(13x^{12}\beta_{13,0} + 11x^{10}t\beta_{13,1} + 9x^8t^2\beta_{13,2} + 7x^6t^3\beta_{13,3} + 5x^4t^4\beta_{13,4} + 3x^2t^5\beta_{13,5}\right. \\
&\qquad \left. + t^6\beta_{13,6}\right) + \\
&A_{15}\left(15x^{14}\beta_{15,0} + 13x^{12}t\beta_{15,1} + 11x^{10}t^2\beta_{15,2} + 9x^8t^3\beta_{15,3} + 7x^6t^4\beta_{15,4} + 5x^4t^5\beta_{15,5}\right. \\
&\qquad \left. + 3x^2t^6\beta_{15,6} + t^7\beta_{15,7}\right) + \cdots
\end{aligned}$$

$$\ldots + A_{2k+1}\left((2k+1)x^{2k}\beta_{2k+1,0} + (2k-1)x^{2k-2}t\beta_{2k+1,1} + \cdots + t^k\beta_{2k+1,k}\right) + \cdots + \quad (19)$$

Taking k times derivatives from both sides of expression (16) we get A_{2n+1} coefficients as following

Ъ $\partial u/\partial x$-| |$_1$$(x = 0) = A_1(10 -$ "

yields

$$\sum_{n=0}^{k} A_{2n+1} t^n \beta_{2n+1,n} = \sum_{n=0}^{k} \frac{\psi^{(n)}(0)}{n!} t^n \tag{20}$$

$$A_{2n+1} = \frac{\psi^{(n)}(0)}{n!\,\beta_{2n+1,n}} \equiv \psi^{(n)}(0) 2^{2n} \tag{21}$$

To find the remaining unknown coefficients A_{2n} we use multinomial coefficients of Newton's Polynomial.

Newton's Polynomial

$$(x_1 + x_2 + \cdots + x_{k+1})^n = \sum_{s_1+s_2+\cdots+s_{k+1}=n} \binom{n}{s_1, s_2, \ldots, s_{k+1}} \prod_{1 \le t \le k+1} x_t^s \tag{22}$$

$$where \binom{n}{s_1, s_2, \ldots, s_{k+1}} = \frac{n!}{s!\,s_2! \ldots s_{k+1}!}$$

is a multinomial coefficient

In our case where

$$x_1 + x_2 + \cdots + x_{k+1} = \alpha(t) \equiv \sum_{n=0}^{k} \alpha_{n+1} t^{n+1} \tag{23}$$

we have $(\alpha_1 t + \alpha_2 t^2 + \cdots + \alpha_{k+1} t^{k+1})^n =$

$$= \sum_{s_1+s_2+\cdots+s_{k+1}=n} \binom{n}{s_1, s_2, \ldots, s_{k+1}} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_{k+1}^{s_{k+1}} t^{s_1+2s_2+\cdots+(k+1)s_{k+1}} \tag{24}$$

where

$$\binom{n}{s_1, s_2, \ldots, s_{k+1}} \alpha_1^{s_1} \alpha_2^{s_2} \ldots \alpha_{k+1}^{s_{k+1}} t^{s_1+2s_2+\cdots+(k+1)s_{k+1}} \tag{25}$$

isa multinomial coefficient in our case.

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} =$$

$$\left.\frac{\partial u}{\partial x}\right|_{x=\alpha(t)} =$$

$$= \sum_{n=0}^{\infty} \left[A_{2n} \sum_{m=0}^{n} (2n-2m) x^{2n-2m-1} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1) x^{2n-2m} t^m \beta_{2n+1,m} \right] \equiv$$

$$\equiv \sum_{n=0}^{\infty} \left[A_{2n} \sum_{m=0}^{n} (2n-2m) \big(\alpha(t)\big)^{2n-2m-1} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1) \big(\alpha(t)\big)^{2n-2m} t^m \beta_{2n+1,m} \right] \tag{26}$$

We substitute (21) into (22) and get

$$\sum_{n=0}^{\infty} \left[A_{2n} \sum_{m=0}^{n} (2n-2m) \big(\alpha(t)\big)^{2n-2m-1} t^m \beta_{2n,m} + A_{2n+1} \sum_{m=0}^{n} (2n-2m+1) \big(\alpha(t)\big)^{2n-2m} t^m \beta_{2n+1,m} \right] =$$

$$\sum_{n=0}^{\infty}\Big[A_{2n}\sum_{m=0}^{n}(2n-2m)\sum_{s_1+s_2+\dots+s_{k+1}=2n-2m-1}\binom{2n-2m-1}{s_1,s_2,\dots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\dots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\dots+(k+1)s_{k+1}+m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+\dots+p_{k+1}=2n-2m}\binom{2n-2m}{p_1,p_2,\dots,p_{k+1}}\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_{k+1}^{p_{k+1}}t^{p_1+2p_2+\dots+(k+1)p_{k+1}+m}\beta_{2n+1,m}\Big] \quad (27)$$

Since $\varphi(t)$ function is analytic and can be expanded into Maclaurin series we can easily derive recurrent formula for A_{2n} coefficients by taking both sides of expression (27) 2k and 2k+1 times derivatives and equate coefficients of both sides.

$$\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)\big(\alpha(t)\big)^{2n-2m-1}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\big(\alpha(t)\big)^{2n-2m}t^m\beta_{2n+1,m}\right]=$$

$$\sum_{n=0}^{\infty}\Big[A_{2n}\sum_{m=0}^{n}(2n-2m)\sum_{s_1+s_2+\dots+s_{k+1}=2n-2m-1}\binom{2n-2m-1}{s_1,s_2,\dots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\dots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\dots+(k+1)s_{k+1}+m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+\dots+p_{k+1}=2n-2m}\binom{2n-2m}{p_1,p_2,\dots,p_{k+1}}\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_{k+1}^{p_{k+1}}t^{p_1+2p_2+\dots+(k+1)p_{k+1}+m}\beta_{2n+1,m}\Big] \quad (28)$$

$$\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)\big(\alpha(t)\big)^{2n-2m-1}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\big(\alpha(t)\big)^{2n-2m}t^m\beta_{2n+1,m}\right]=$$

$$\sum_{n=0}^{\infty}\Big[A_{2n}\sum_{m=0}^{n}(2n-2m)\sum_{s_1+s_2+\dots+s_{k+1}=2n-2m-1}\binom{2n-2m-1}{s_1,s_2,\dots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\dots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\dots+(k+1)s_{k+1}+m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+\dots+p_{k+1}=2n-2m}\binom{2n-2m}{p_1,p_2,\dots,p_{k+1}}\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_{k+1}^{p_{k+1}}t^{p_1+2p_2+\dots+(k+1)p_{k+1}+m}\beta_{2n+1,m}\Big] \quad (29)$$

$$\sum_{n=0}^{\infty}\left[A_{2n}\sum_{m=0}^{n}(2n-2m)\big(\alpha(t)\big)^{2n-2m-1}t^m\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\big(\alpha(t)\big)^{2n-2m}t^m\beta_{2n+1,m}\right]=$$

$$\sum_{n=0}^{\infty}\Big[A_{2n}\sum_{m=0}^{n}(2n-2m)\sum_{s_1+s_2+\dots+s_{k+1}=2n-2m-1}\binom{2n-2m-1}{s_1,s_2,\dots,s_{k+1}}\alpha_1^{s_1}\alpha_2^{s_2}\dots\alpha_{k+1}^{s_{k+1}}t^{s_1+2s_2+\dots+(k+1)s_{k+1}+m}\beta_{2n,m}+A_{2n+1}\sum_{m=0}^{n}(2n-2m+1)\sum_{p_1+p_2+\dots+p_{k+1}=2n-2m}\binom{2n-2m}{p_1,p_2,\dots,p_{k+1}}\alpha_1^{p_1}\alpha_2^{p_2}\dots\alpha_{k+1}^{p_{k+1}}t^{p_1+2p_2+\dots+(k+1)p_{k+1}+m}\beta_{2n+1,m}\Big] \quad (30)$$